**Calculus III - SM221**
**Test 4 Review Notes**

## I. Vector Fields

A. Assign vectors to points in $R^2$ or $R^3$

B. $\vec{\mathbf{F}} = \nabla f$

1. $\vec{\mathbf{F}}$ is a conservative vector field
2. $f$ is the potential function of $\vec{\mathbf{F}}$

## II. Line Integrals

A. For piecewise smooth curve C given by $\vec{\mathbf{r}}(t) = \langle x(t), y(t) \rangle$ on $a \le t \le b$:

$$\int_C f(x,y)ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b f(\vec{\mathbf{r}}(t))\left|\vec{\mathbf{r}}'(t)\right|dt$$

B. For piecewise smooth curve C given by $\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t) \rangle$ on $a \le t \le b$

$$\int_C f(x,y,z)ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \int_a^b f(\vec{\mathbf{r}}(t))\left|\vec{\mathbf{r}}'(t)\right|dt$$

C. For a continuous vector field $\vec{\mathbf{F}}$ defined on C given by $\vec{\mathbf{r}}(t)$ on $a \le t \le b$

$$\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}} \bullet \hat{\mathbf{T}}ds = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \bullet \vec{\mathbf{r}}'(t)dt$$

1. Work done by $\vec{\mathbf{F}}$ on a particle traversing C is given by $W = \int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$

## III. Fundamental Theorem for Line Integrals

If $\vec{\mathbf{F}} = \langle P, Q \rangle$ is a *conservative* vector field defined on C given by $\vec{\mathbf{r}}(t)$ on $a \le t \le b$,

$$\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_C \nabla f \bullet d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$$

$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = 0$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

## IV. Green's Theorem

A. If $\vec{\mathbf{F}} = \langle P, Q \rangle$ is defined on D bounded by simple C given by $\vec{\mathbf{r}}(t)$ on $a \le t \le b$,

$$\oint_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$$

B. Surface Area using Green's Theorem

$$A(D) = \oint_C x\,dy = -\oint_C y\,dx = \tfrac{1}{2}\oint_C x\,dy - y\,dx$$

## V. Curl and Divergence

A. Curl

1. $\operatorname{curl} \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

2. $\operatorname{curl} \vec{\mathbf{F}} = \operatorname{curl} \nabla f = 0 \quad \Leftrightarrow \quad \vec{\mathbf{F}}$ is conservative

B. Divergence

1. $\operatorname{div} \vec{\mathbf{F}} = \nabla \bullet \vec{\mathbf{F}} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z}$

2. For vector field $\vec{\mathbf{F}}$

   $\operatorname{div} \operatorname{curl} \vec{\mathbf{F}} = 0$

**VI. Surface Integrals**

A. Scalar Functions

1. $\iint_S f(x,y,z)\,dS = \iint_D f(\vec{\mathbf{r}}(u,v))\left|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\right| dA$

2. Good example is finding the total charge on a surface $S$ given charge density $f(x,y,z)$

B. Vector Fields

1. Also known as ***Flux Integral***
2. $\iint_S \vec{\mathbf{F}}(x,y,z)\,dS = \iint_S \vec{\mathbf{F}} \bullet \hat{\mathbf{n}}\,dS = \iint_D \vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v)) \bullet \left(\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\right) dA$

   a. $\vec{\mathbf{n}} = \dfrac{\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v}{\left|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\right|}$

**VII. Stokes' Theorem**

A. $\iint_S curl\vec{\mathbf{F}} \cdot dS = \oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$

1. Surface $S$ is bordered by closed curve $C$
2. Note that you can use a simpler surface with the same boundary curve

**VIII. Divergence Theorem**

A. $\iint_S \vec{\mathbf{F}} \cdot dS = \iiint_E div\vec{\mathbf{F}}\,dV$

1. Simple surface $S$ encloses region $E$
2. Note that if $div\vec{\mathbf{F}}$ is a constant, then $\iiint_E div\vec{\mathbf{F}}\,dV = K \cdot \iiint_E dV = KV$ where V is the volume of the region enclosed by $S$.